Introduction to Cosmology

Lecture 2: Distances in the universe

Plan

1. The horizon problem
2. Luminosity and angular distances
3. Dynamics of the scale factor
4. Friedmann equation

1. The horizon problem

Redshift of a far away galaxy well-defined, but not its distance. We need $a(t)$! Use preceding explicit model for $a(t)$

Far away in distance means far away in time. Photon emitted by far away galaxy at time t, received by us at time $t = t_0$, comoving distance

$$d_{\mathrm{com}} = c \int_t^{t_0} \frac{\mathrm{d}t'}{a(t')}$$

Use redshift z as integration variable

$$\frac{\mathrm{d}t}{a(t)} = \frac{\mathrm{d}z}{H_0} (1+z)^{-3/2}$$

(Within the specific model!). Then distance d (note $a(t_0) = 1$)

$$d = a(t_0)d_{\rm com}(z) = \frac{ca(t_0)}{H_0}\int_1^z {\rm d}z(1+z)^{-3/2}$$
$$= \frac{2ca(t_0)}{H_0}\left[1-(1+z)^{-1/2}\right]$$

Photon emitted at the Big Bang, $z \to \infty$, defines the particle horizon or simply horizon

$$d_{\rm hor}(t_0) = 2c/H_0$$

to be contrasted with naive estimate $d_{\rm hor}(t_0) = c/H_0$.

Horizon of a galaxy at $\overline{z}$

$$r_{\rm com}(\overline{z}) = \frac{2c}{H_0}(1+\overline{z})^{-1/2}$$

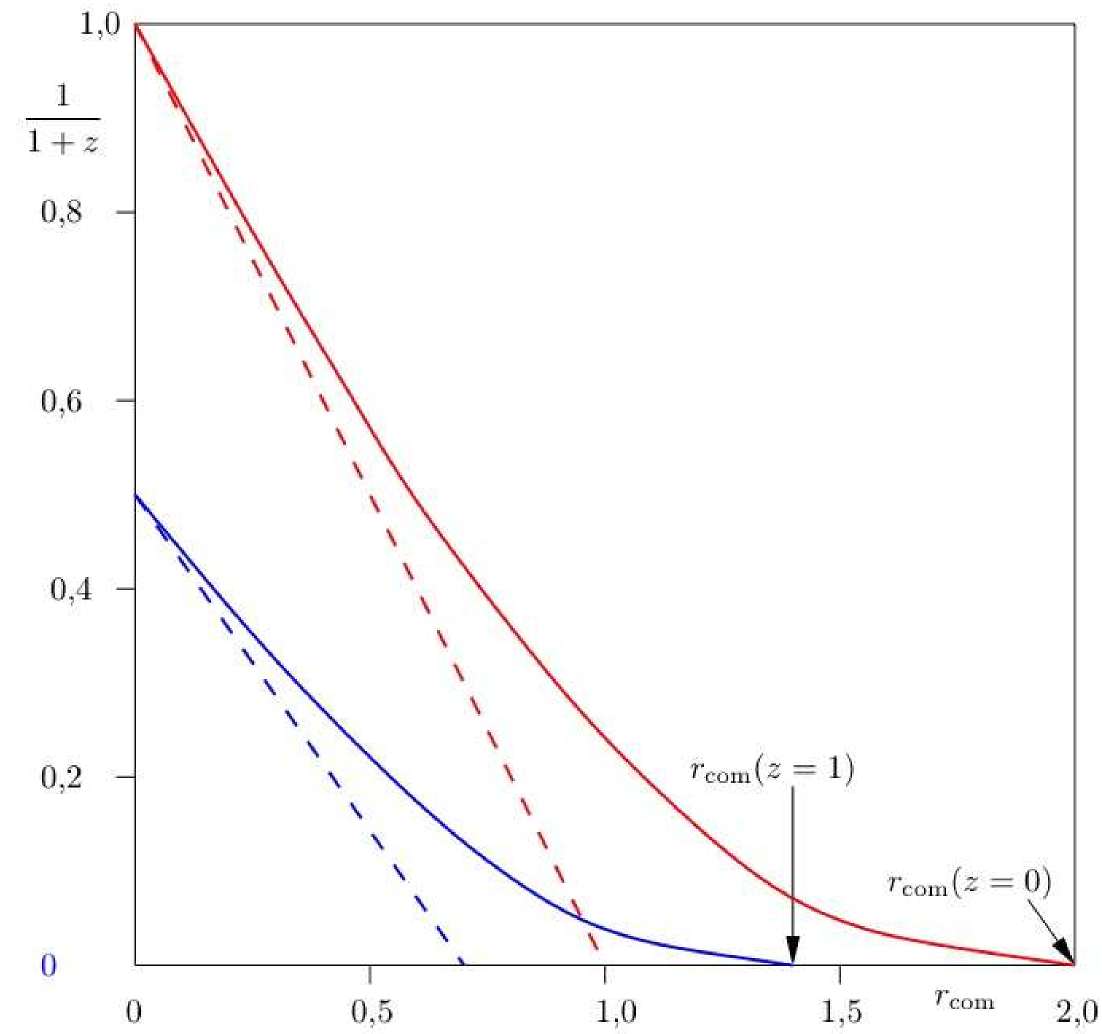

Figure 1: Horizons at $z = 0$ (red) and $z = \overline{z}$ (blue), horizontal units c/H_0. Horizon = infinite redshift.

Comoving scales within horizon today beyond horizon at $\overline{z}$

Actually we cannot receive photons emitted at $t = 0$, universe opaque to photons until their decoupling at $t = t_{\rm dec} \simeq 380\,000$ years after the Big Bang. Best estimates of horizon with more realistic model

$$d_{\rm hor}(t_0) \simeq 3\,ct_0 \simeq \frac{3c}{H_0} \simeq 45 \text{ billions l} - \text{y}$$

The horizon problem. Photons decouple at

$$z_{\rm dec} = a(t_0)/a(t_{\rm dec}) \simeq 1\,100$$

The horizon at $t = t_{\rm dec}$ is $\simeq cH_0^{-1}(1 + z_{\rm dec})^{-1/2}$, so seen by us under an angle angle

$$\theta \simeq (1 + z_{\rm dec})^{-1/2} \sim 1 \text{ degree}$$

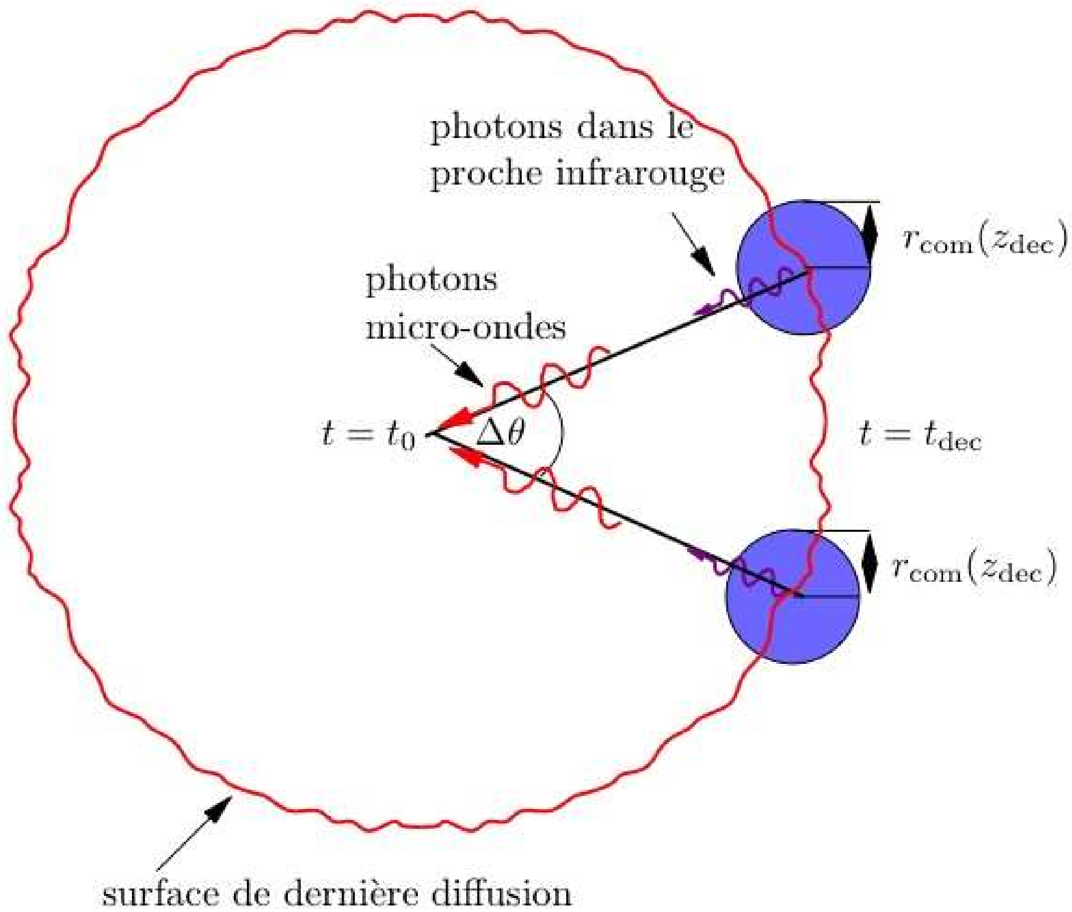

Figure 2: The horizon problem. Photons emitted at wavelength $\lambda \simeq 1.7\,\mu$m (infrared) and received at $\lambda \simeq 1.9\,$mm (microwave). Surface of last scattering = effective limit of observable universe

2. Luminosity distance and angular distance

The flux-luminosity formula needs a modification. Define d_{eff} from flux-luminosity formula. But

1. Measured photon frequency ν_0, and hence energy $E_\gamma = 2\pi\hbar\nu$ smaller than emitted frequency ν_e because of cosmological redshift

$$\nu_0 = \frac{\nu_e}{(1+z)}$$

2. Interval Δt_e between emission times of two photons larger than interval between reception times

$$\Delta t_0 = \Delta t_e (1+z)$$

Luminosity distance d_L

$$\frac{f}{L} = \frac{1}{4\pi d_{\text{eff}}^2} \frac{1}{(1+z)^2} = \frac{1}{4\pi d_L^2}$$

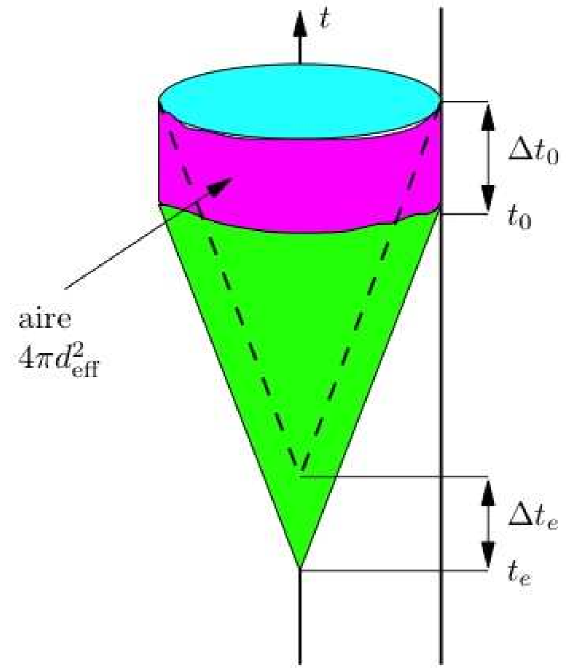

Figure 3: The flux-luminosity relation in a two-dimensional space-time

Angular distance d_A. Photon propagation at constant (θ, φ). Object size ΔL, emission at comoving distance $r_{\rm com}$, seen under angle $\Delta\theta$

$$\Delta L = a(t) r_{\rm com} \Delta\theta$$

The angle $\Delta\theta$ does not vary

$$\Delta\theta = \frac{\Delta L}{a(t) r_{\rm com}} = \frac{\Delta L (1+z)}{a(t_0) r_{\rm com}} = \frac{\Delta L}{d_A}$$

Relation between d_L and d_A

$$d_A = a(t_0) r_{\rm com} (1+z)^{-1} = d_L (1+z)^{-2}$$

If space is curved, for the same angular aperture $\Delta\theta$

$$\Delta L_{\rm curv.>0} < \Delta L_{\rm curv.=0} < \Delta L_{\rm curv.<0}$$

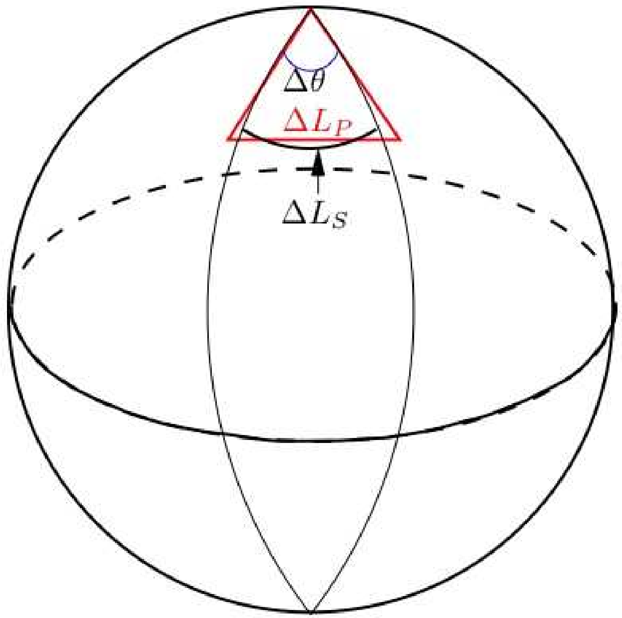

Figure 4: Distances in a curved space

Distance concept ambiguous: depends on the definition and of the expansion of the universe. No universal definition of distance. Use the redshift z!

3. Dynamics of the scale factor

Simple energetic considerations. Universe $\equiv$ ideal fluid of galaxies, no heat transfer, no entropy creation. Standard thermodynamic relation $\mathrm{d}E = T\mathrm{d}S - \mathcal{P}\mathrm{d}V$

$$\delta E = -\mathcal{P}\delta V$$

In general $\delta E > 0$ when $\delta V < 0$ as $\mathcal{P} > 0$

If $\delta E > 0$ when $\delta V > 0$, then $\mathcal{P} < 0$!

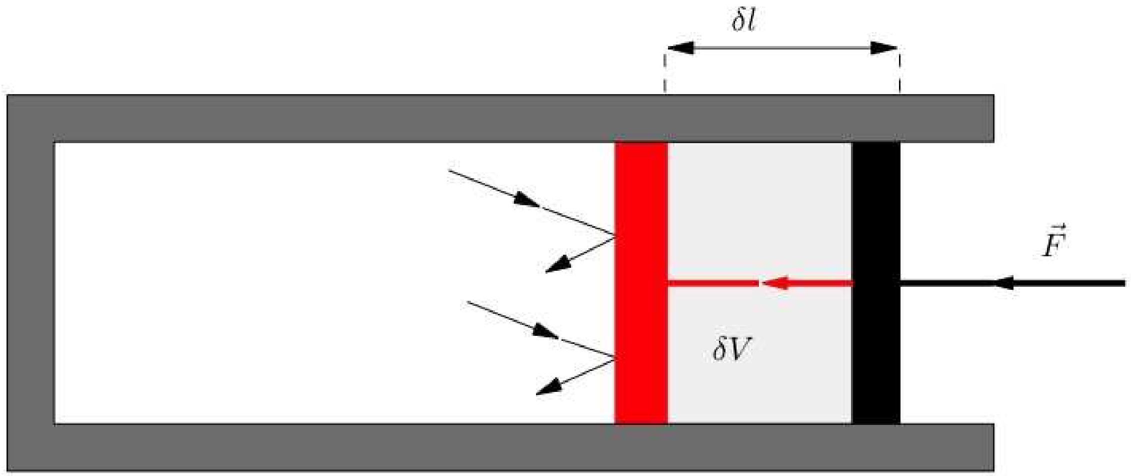

Figure 5: Fluid compression

E increases when V decreases if $\mathcal{P} > 0$. $V_{\rm com}$ covolume containing fixed number of galaxies $V(t) = a^3(t)\, V_{\rm com}$

$$\delta\left[\rho(t)a^3(t)\Delta V_{\rm com}\right] = -\mathcal{P}(t)\,\delta\left[a^3(t)\Delta V_{\rm com}\right]$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\rho(t)a^3(t)\right] = -\mathcal{P}(t)\,\frac{\mathrm{d}}{\mathrm{d}t}\left[a^3(t)\right]$$

There limiting cases

1. Matter dominated universe. Pression negligible w.r.t. mass energy

   $$\rho(t)a^3(t) = \mathrm{cst} \qquad \text{or} \qquad \rho(t) = \rho(t_0)(1+z)^3$$

   Energy density $\times$ volume $=$ constant

2. Radiation dominated universe. Then (black body radiation) $\mathcal{P} = \rho/3$ and $\rho \propto T^4$, valid for any gas of ultra-

relativistic particles

$$\frac{\mathrm{d}}{\mathrm{d}t}[\rho(t)a^4(t)] = 0$$

Since $\rho \propto T^4$ scale factor $a(t) \propto 1/T$

$$\begin{aligned} \rho(t) &= \rho(t_0)\left[\frac{a(t_0)}{a(t)}\right]^4 = \rho(t_0)(1+z)^4 \\ T(t) &= T(t_0)\left[\frac{a(t_0)}{a(t)}\right] = T(t_0)(1+z) \end{aligned}$$

CMB temperature $\sim 3\,\mathrm{K}$, decoupling temperature $\sim 3000\,\mathrm{K}$, $z_{\mathrm{dec}} \sim 10^3$

3. Universe dominated by dark energy Assumption: dark energy = vacuum energy, density ρ_v time independent, no dilution, so that $\mathcal{P} = -\rho_v$ Unfortunately QFT gives

10^{120} time the observed density. Other possibility: cosmological constant

Friedmann equation

From Einstein equation, using $T^\mu{}_\nu$ for a perfect fluid

$$T^\mu{}_\nu = (\mathcal{P} + \rho) u^\mu u_\nu - \mathcal{P} \delta^\mu{}_\nu \qquad u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$$

Proof: fluid rest frame.

Using general metric, with possible space curvature

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 - \frac{8\pi}{3} G\rho(t) = -\frac{kc^2}{R^2 a^2(t)} \qquad k = 1, 0, -1$$

R = radius of curvature, $k = +1$: positive curvature, $k = 0$: flat space ($R \to \infty$), $k = -1$ negative curvature. Critical density

$$\rho_0 = \rho_c = \frac{3H_0^2}{8\pi G}$$

1. $\rho_0 < \rho_c$: open universe, expansion does not stop
2. $\rho_0 = \rho_c$: open universe, expansion does not stop
3. $\rho_0 > \rho_c$: closed universe, expansion stops and Big Crunch

However does not tell anything on the topology of space!

However strict connection between density and fate of the universe valid only if $\rho_v = 0$

One usually defines the ratios at $t = t_0$

$$\Omega_m = \frac{\rho_{m0}}{\rho_c} \qquad \Omega_r = \frac{\rho_{r0}}{\rho_c} \qquad \Omega_v = \frac{\rho_{v0}}{\rho_c}$$

and $\Omega_m + \Omega_r + \Omega_v = 1$ for a flat universe. Time derivative of Friedmann equation multiplied by a^2

$$\ddot{a} - \frac{4\pi}{3} G\dot{\rho}\frac{a^2}{\dot{a}} - \frac{8\pi}{3} G\rho a = 0$$

so that

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(3\mathcal{P} + \rho)$$

$\ddot{a} < 0$ if $(3\mathcal{P} + \rho) > 0$. Acceleration of expansion implies negative pressures! Limiting cases

1. Matter dominated universe: $\Omega_m = 1, \Omega_r = \Omega_v = 0$. $a(t) \propto t^{2/3}$, $t_0 = 2/(3H_0)$.
2. Radiation dominated universe: $\Omega_m = \Omega_v = 0, \Omega_r = 1$. Then $a(t) \propto t^{1/2}$ At sufficiently early times, universe radiation dominated.
3. Vacuum dominated universe: $\Omega_m = \Omega_r = 0, \Omega_v = 1$. Then

$$a(t) = a(t_0)\, \mathrm{e}^{H(t-t_0)}$$

   At late enough times, universe dominated by vacuum energy.

Parameters from Planck

$$\Omega_m \simeq 0.3,\ \Omega_v \simeq 0.7,\ \Omega_r \simeq 0$$

Matter and radiation roughly equivalent for $z \simeq 3000$. Matter and vacuum energy roughly equivalent for $t \simeq 7 \times 10^9$ years

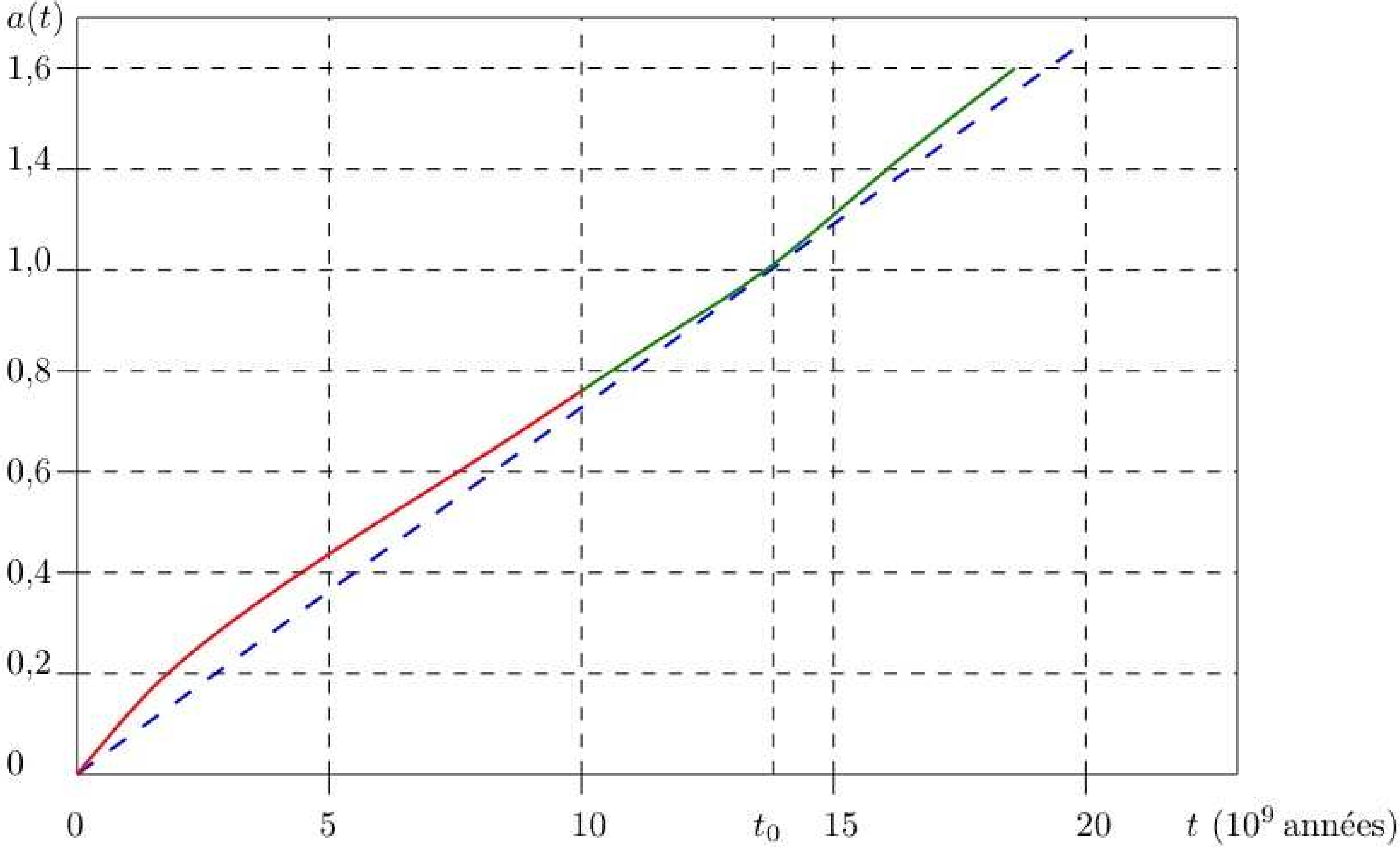

Figure 6: A semi-realistic graph for $a(t)$